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Homotopy, monopoles and 't Hooft tensor in QCD with generic gauge group

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Abstract: We study monopoles and corresponding 't Hooft tensor in QCD with a generic compact gauge group. This issue is relevant to the understanding of the color confinement in terms of dual symmetry.

Keywords: Confinement, Duality in Gauge Field Theories, Lattice Gauge Field Theories.

JHEP10(2008)096

Contents

JHEP10(2008)096

1. Introduction

Any explanation of color confinement in terms of a dual symmetry, requires the existence of field configurations with non trivial spatial homotopy Π_2. This amounts to extend the formulation of the theory to a spacetime with an arbitrary but finite number of line-like singularities (monopoles) [1].

A prototype example of such configuration is the 't Hooft - Polyakov monopole [2, 3] in the SO(3) gauge theory interacting with a Higgs scalar in the adjoint color representation. It is a static soliton solution made stable by its non trivial homotopy.

In the "hedgehog" gauge the i-th color component of the Higgs field $\phi(r) = \phi^i(r)\sigma_i$ at large distances has the form

$$\phi^i \simeq \frac{r^i}{|\vec{r}|} \tag{1.1}$$

and is a mapping of the sphere S_2 at spatial infinity on $\mathrm{SO}(3)/\mathrm{U}(1)$, with non trivial homotopy. In the unitary gauge, where $\frac{\phi^i}{|\phi|} = \delta^i_3\,\sigma_3$ is diagonal, a line singularity appears starting from the location of the monopole.

The Abelian field strength of the residual $\mathrm{U}(1)$ symmetry in the unitary gauge is given by

$$F_{\mu\nu} = \partial_\mu A^3_\nu - \partial_\nu A^3_\mu \tag{1.2}$$

The monopole configuration has zero electric field ($F_{0i} = 0$) and the magnetic field $H_i = \frac{1}{2}\,\epsilon_{ijk}F_{jk}$ is the field of a Dirac monopole of charge 2

$$\vec{H} = \frac{1}{g}\frac{\vec{r}}{4\pi r^3} + \text{Dirac String} \tag{1.3}$$

In a compact formulation, like is lattice, the Dirac string is invisible and a violation of Bianchi identity occurs

$$\vec{\nabla} \cdot \vec{H} = \frac{1}{g}\, \delta^3(x) \tag{1.4}$$

More formally, one can define a covariant field strength $F_{\mu\nu}$ which coincides, in the unitary gauge, with the abelian field strength of the residual symmetry [2]

$$F_{\mu\nu} = Tr(\hat{\phi}\, G_{\mu\nu}) - \frac{i}{g}\, Tr\left(\hat{\phi}\, [D_\mu \hat{\phi}, D_\nu \hat{\phi}]\right) \tag{1.5}$$

Here

$$\hat{\phi} = \sum \hat{\phi}^a T^a \qquad\qquad G_{\mu\nu} = \sum G^a_{\mu\nu} T^a$$

$$\hat{\phi}_a = \frac{\phi_a}{|\,\phi_a|} \qquad\qquad D_\mu \hat{\phi} = \partial_\mu \hat{\phi} + ig[A_\mu, \hat{\phi}]$$

T^a are the group generators with normalization $Tr(T^a T^b) = \frac{1}{2}\,\delta^{ab}$. $F_{\mu\nu}$ is known as 't Hooft tensor. A magnetic current can be defined as

$$j_\nu = \partial^\mu \widetilde{F}_{\mu\nu} \tag{1.6}$$

where $\tilde{F}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma} F_{\rho\sigma}$. A non zero value of it signals the violation of Bianchi identities. Furthermore, the current defined in eq. (1.6) is identically conserved

$$\partial^\nu j_\nu = 0 \tag{1.7}$$

The main feature of eq. (1.5) is that linear and bilinear terms in A_μ, A_ν cancel and one has identically

$$F_{\mu\nu} = Tr\left(\partial_\mu(\hat{\phi} A_\nu) - \partial_\mu(\hat{\phi} A_\nu) - \frac{i}{g}\, \hat{\phi}[\partial_\mu \hat{\phi}, \partial_\nu \hat{\phi}]\right) \tag{1.8}$$

In the unitary gauge, where $\hat{\phi} = (0, 0, 1)$ and $\partial_\mu \hat{\phi} = 0$, it reduces to eq. (1.2).
In a theory with no Higgs field a 't Hooft tensor can be defined by choosing

$$\phi = U(x) \sigma_3 U(x)^\dagger \tag{1.9}$$

with $U(x)$ any element of the group, for example the parallel transport to x from a fixed arbitrary point at infinity. $U(x)^\dagger$ is the gauge transformation to the unitary gauge.

Again a conserved magnetic current, identifying a dual symmetry, can be defined. In principle any field ϕ in the adjoint representation can be used as effective Higgs: all of them have the form of eq. (1.9) except for a finite number of singularities and differ from each other by a gauge transformation defined everywhere except at singularities.

The generalization to SU(N) is designed in ref. [2] and developed in detail in ref. [4]. The strategy is to ask what fields ϕ would allow the definition of 't Hooft tensor, with the cancelations bringing from eq. (1.5) to eq. (1.8), so that it becomes the abelian residual field strength in the unitary gauge.

JHEP10(2008)096

The answer is that there are $N-1$ such fields (as many as the rank of the group), one for each fundamental weight. Explicitly

$$\phi^a(x) = U(x)\phi_0^a U^\dagger(x) \tag{1.10}$$

with ϕ_0^a the fundamental weight

$$\phi_0^a = \frac{1}{N}\,\mathrm{diag}\,(\underbrace{N-a,\ldots,N-a}_{a},\underbrace{-a,\,\ldots,\,-a}_{N-a}) \tag{1.11}$$

$(a = 1 \ldots N-1)$. The invariance group of ϕ_0^a is $\mathrm{SU}(a)\times\mathrm{SU}(N-a)\times\mathrm{U}(1)$ and the quotient group $\frac{\mathrm{SU}(N)}{\mathrm{SU}(a)\times\mathrm{SU}(N-a)\times\mathrm{U}(1)}$ has non trivial homotopy

$$\Pi_2\left(\frac{\mathrm{SU}(N)}{\mathrm{SU}(a)\times\mathrm{SU}(N-a)\times\mathrm{U}(1)}\right) = \mathbf{Z}$$

(for a more precise formulation see section 3 below). There exist $N-1$ monopole species for $\mathrm{SU}(N)$, one for each a.

To connect with the approach of ref. [5], if $\psi(x)$ is a generic hermitian operator in the adjoint representation, it can be diagonalized to $\psi_0(x)$. Since the maximal weights are a complete set of traceless $N\times N$ diagonal matrices, one has

$$\psi_0(x) = \sum_{a=1}^{N-1} c_a(x)\phi_0^a \tag{1.12}$$

and

$$\psi(x) = \sum_{a=1}^{N-1} c_a(x)\phi^a(x)\,. \tag{1.13}$$

$c_a(x)$ is the difference of two subsequent eigenvalues of ψ

$$c_a(x) = \psi_0(x)_a^a - \psi_0(x)_{a+1}^{a+1}$$

and is equal to zero at the sites where two eigenvalues coincide and there a singularity appears in the unitary gauge, corresponding to a monopole of species a sitting at x.

Recently some special groups like G_2 and F_4 became of interest, since they have no center and seem to confine [6], in contrast with the idea that center vortices could be the configurations responsible for confinement [7]. It is thus interesting to investigate monopole condensation in these systems.

However, for the group G_2 and F_4 it proves impossible to construct a 't Hooft tensor of the form of eq. (1.5): no solution exists for ϕ^a, such that eq. (1.5), (1.8) are valid. Still, as we shall see in the following, there are monopoles in these theories and it is possible to define magnetic conserved currents. The approach sketched above, which works for SU(2) and $\mathrm{SU}(N)$, has to be modified for a more general construction of a 't Hooft like tensor. We approach and solve this problem in the present paper.

We will consider theories like QCD (gluons plus at most quarks) with a generic compact gauge group and no Higgs fields: we shall only use a Higgs field in the adjoint representation as a tool to classify the dual symmetry. In particular, we shall not consider supersymmetric extensions.

JHEP10(2008)096

2. Monopoles

Let G be a gauge group, which we shall assume to be compact and simple. To define a monopole current we have to isolate an SU(2) subgroup, and break it to its third component, say T_3. This will be done by some "Higgs field" ϕ in the adjoint representation.

Our notation is the familiar one (see e.g. [8, 9]). There are r commuting generators of G (r=rank of group) which we shall denote as H_i ($i = 1, \ldots, r$). The other generators occur in pairs with opposite values of Cartan eigenvalues:

$$\begin{aligned} &[H_i, H_j] = 0 & &[H_i, E_{\pm\vec{\alpha}}] = \pm\alpha_i E_{\pm\vec{\alpha}} \\ &[E_{\vec{\alpha}}, E_{\vec{\beta}}] = N_{\vec{\alpha},\vec{\beta}} E_{\vec{\alpha}+\vec{\beta}} & &[E_{\vec{\alpha}}, E_{-\vec{\alpha}}] = \alpha_i H_i \end{aligned} \tag{2.1}$$

where $\vec{\alpha} = (\alpha_1, \ldots, \alpha_r)$ and $N_{\alpha,\beta} \neq 0$ only if $\vec{\alpha} + \vec{\beta}$ is a root. The root $\vec{\alpha}$ can be taken positive ($-\vec{\alpha}$ negative). By definition, a root is positive if its first nonzero component is positive: either $\vec{\alpha}$ or $-\vec{\alpha}$ is positive. Of course the choice is conventional and also depends on the choice for the order of components. A positive root is called simple if it cannot be written as the sum of two other positive roots.

The way to associate an $su(2)$ algebra to each root is a trivial renormalization of $E_{\pm\vec{\alpha}}$. Defining

$$T^{\alpha}_{\pm} = \sqrt{\frac{2}{(\vec{\alpha}\cdot\vec{\alpha})}} E_{\pm\vec{\alpha}} \quad T^{\alpha}_3 = \frac{\vec{\alpha}\cdot\vec{H}}{(\vec{\alpha}\cdot\vec{\alpha})}$$

we have

$$[T^{\alpha}_3, T^{\alpha}_{\pm}] = \pm T^{\alpha}_{\pm} \quad [T^{\alpha}_+, T^{\alpha}_-] = 2\, T^{\alpha}_3$$

A Weyl transformation is an invariance transformation of the algebra which permutes the roots [8, 9]. It can be proved that any root can be made a simple root by a Weyl transformation ([8] III.10 pg.51). Furthermore it can also be proved that the Weyl transformations are induced by transformations of the group G ([9] VIII.8 pg.193). If the Higgs potential is invariant under G, we can then consider without loss of generality only the SU(2) subgroups related to the simple roots.

A *vev* of the field ϕ proportional to any of the fundamental weights μ^i, $i = (1, \ldots, r)$, corresponding to the i-th simple root, identifies a monopole.[1] Indeed recall that

$$\mu^i = \vec{c}^{\,i} \cdot \vec{H} \quad [\mu^i, T^j_{\pm}] = \pm\, \vec{c}_i \cdot \vec{\alpha}_j\, T^j_{\pm} = \pm\, \delta_{ij}\, T^j_{\pm}$$

Taking

$$\mu_i = T^i_3 + (\mu^i - T^i_3) \tag{2.2}$$

the last term commutes with $T^i_{\pm}$, T^i_3

$$[\mu^i, T^j_{\pm}] = \pm\,\delta_{ij}\, T^j_{\pm} \qquad [\mu^i, T^j_3] = 0 \qquad [T^i_3, T^j_{\pm}] = \pm\,\delta_{ij}\, T^j_{\pm} \tag{2.3}$$

[1]This kind of breaking is called *maximal* and identifies r magnetic charges, one for each fundamental weight. Configurations carrying a non zero value of more than one of this charge (non maximal breaking) exist [10], but they don't add any new information concerning the symmetry.

JHEP10(2008)096

The little group of ϕ^i, $\tilde{H}$, is the product of the U(1) generated by μ^i times a group H which has as Dynkin diagram the diagram (connected or not connected) obtained by erasing from the diagram of G the root α_i and the links which connect it to the rest (Levi subgroup):

$$\tilde{H} = H \times \mathrm{U}(1) \tag{2.4}$$

Indeed $\phi = \mu^i$ commutes with all the simple roots different from α_i and of course with the H_i.

The 't Hooft tensor will be, by definition, a gauge invariant tensor which coincides with

$$F^i_{\mu\nu} = \partial_\mu A^3_\nu - \partial_\nu A^3_\mu \tag{2.5}$$

in the unitary gauge in which ϕ^i is diagonal. The index 3 labels the component along T^i_3, the diagonal generator of the broken SU(2). As we did for the case of SU(2), we will define r magnetic currents j^i_μ as

$$j^i_\mu = \partial_\nu \widetilde{F}^i_{\mu\nu} \tag{2.6}$$

$$\partial^\mu j^i_\mu = 0 \tag{2.7}$$

and the corresponding magnetic charges

$$Q^i = \int d^3x j^i_0(\vec{x}, t). \tag{2.8}$$

The index i runs from 1 to r, the rank of the group . The analogue for this breaking of the 't Hooft-Polyakov solution [2, 3], in presence of a Higgs field, would be

$$A^i_k = A^m_k(\vec{r}) T^i_m, \qquad \phi(\vec{r})^i = \chi^m(\vec{r}) T^i_m + (\mu^i - T^i_3) \tag{2.9}$$

where

$$\begin{aligned} A^m_k(\vec{r}) &= g(r)\epsilon_{mkj}\frac{r^j}{r^2}, & \chi^m(\vec{r}) &= \frac{r^m}{r}\chi(r) \\ g(\infty) &= 1 & \chi(\infty) &= 1 \end{aligned} \tag{2.10}$$

It is a solution like that of ref. [2, 3] inside the SU(2) subgroup generated by $T^i_\pm$, T^i_3. The index m indicates color, while the indices k, j space directions and the index i refers to the simple root chosen. It is straightforward to verify that this monopole is charged under the magnetic U(1) generated by T^i_3. A complete geometric classification of the configurations in term of magnetic charges will be given in the next section.

3. Monopole charge and homotopy

Monopole configurations can be classified in terms of the second homotopy group $\Pi_2(G/\tilde{H})$. In the following we will use the relationship [11]

$$\pi_2(G/\tilde{H}) \simeq \ker[\pi_1(\tilde{H}) \to \pi_1(G)] \tag{3.1}$$

and we will compute $\pi_1(\tilde{H})$ following the formulation of [12]. We consider two gauge fields, respectively defined on north ($0 \leq \theta \leq \pi/2$) and south ($\pi/2 \leq \theta \leq \pi$) hemisphere, of the form

$$A^{\pm}_{\varphi} = \pm g\, T_3 (1 \mp \cos\theta), \tag{3.2}$$

with φ the azimuthal direction. A^{+}_{φ} is defined on the north hemisphere and A^{-}_{φ} in the south one. T_3 is the third component of the broken SU(2).

On the equator this two solutions must be transformed one into each other by a gauge transformation of the form

$$\Omega = \exp(i\, 2\, e\, g\, T_3\, \varphi) \tag{3.3}$$

which is single-valued if

$$\exp(i\, 4\, \pi\, e\, g\, T_3) = 1 \tag{3.4}$$

In the simple case of $G = \mathrm{SU}(2)$ and $\tilde{H} = \mathrm{U}(1)$, eq. (3.4) gives the Dirac quantization condition

$$g = \frac{n}{2e} \tag{3.5}$$

Monopoles are identified by an integer n, the winding number on $\tilde{H} = \mathrm{U}(1)$ group. Indeed

$$\Pi_2(\mathrm{SU}(2)/\,\mathrm{U}(1)) = \Pi_1(\mathrm{U}(1)) = \mathbf{Z} \tag{3.6}$$

For a generic gauge group G the discussion turns out to be more involved, since the analysis of $\Pi_2(G/\tilde{H})$ is related to the global (topological) structure of G and $\tilde{H}$ which in general cannot be inferred from their Lie algebras.

In general

$$\tilde{H} = \frac{H \times \mathrm{U}(1)}{Z} \tag{3.7}$$

where Z is a subgroup of the center of $H \times \mathrm{U}(1)$. This happens when the identity of G can be written not only as the identity of H times the identity of $\mathrm{U}(1)$ but also as an element of $\mathrm{U}(1)$ times a non trivial element z of H. Since $\mathrm{U}(1)$ commutes with H, z must commute with all elements of H and hence it belongs to its center. Mathematically speaking, Z is the kernel of the map $\Phi :\ H \times \mathrm{U}(1) \to G$.

For example, for $G = \mathrm{SU}(N)$, one can check that the residual invariance group is

$$\tilde{H} = \frac{\mathrm{SU}(a) \times \mathrm{SU}(N-a) \times \mathrm{U}(1)}{\mathbf{Z}_k} \tag{3.8}$$

where k is the *mcm* between a and $N-a$. The third component of the broken SU(2) is

$$T_3 = \mathrm{diag}(0, \ldots, 1, -1, \ldots, 0), \tag{3.9}$$

so that the usual Dirac quantization $g = \frac{n}{2e}$, in terms of the minimal electric charge [13, 14], follows from eq. (3.4). Monopole configurations are labeled by an integer n.

To see the correspondence between the $\mathrm{U}(1)$ magnetic charges and the non-contractible loops on $\tilde{H}$, we substitute the value of eg as determined from eq. (3.4) into eq. (3.3) obtaining

$$\Omega = \exp(i\ n\, T_3\, \varphi) \tag{3.10}$$

JHEP10(2008)096

Magnetic charges (with various n) are associated to loops that wind n-times on magnetic U(1), the subgroup generated by T_3.

From the point of view of the $\tilde{H}$ group, every monopole charge is in one-to-one correspondence with a loop that starts from identity, moves inside the U(1) to an element of the center of $\mathrm{SU}(a)\times\mathrm{SU}(N-a)$ and comes back to identity along a path into $\mathrm{SU}(a)\times\mathrm{SU}(N-a)$.

Algebraically one can write (see eq. (2.2)) [13, 14]:

$$T_3 = \phi + h \tag{3.11}$$

with

$$\phi = \mathrm{diag}\left(\frac{1}{a},\ \dots\ \frac{1}{a},\ -\frac{1}{N-a},\ \dots\ -\frac{1}{N-a}\right) \tag{3.12}$$

$$h = \mathrm{diag}\left(-\frac{1}{a},\ \dots\ \frac{a-1}{a},\ \frac{a+1-N}{N-a},\ \dots\ \frac{1}{N-a}\right) \tag{3.13}$$

where ϕ is the effective Higgs and h is an element of the Cartan subalgebra of H. By use of formula (3.10), we easily recognize that the loops in the U(1) with winding number L correspond to magnetic charges $n = L\,k$ since, for $\varphi = 2\pi$, $e^{i\,2\pi\,\phi\,L\,k} = I$. Charges of the form

$$n = q + L\,k \qquad q \neq 0 \tag{3.14}$$

are associated to loops that go inside U(1) from identity to

$$\exp\left(i\phi\,2\pi\,q\right) = \exp\left(\frac{2\pi i q}{a}\ \dots\ \frac{2\pi i q}{a},\ -\frac{2\pi i q}{N-a}\ \dots\ -\frac{2\pi i q}{N-a}\right), \tag{3.15}$$

an element of the center of $\mathrm{SU}(a)\times\mathrm{SU}(N-a)$, and come back through the $\mathrm{SU}(a)\times\mathrm{SU}(N-a)$ part (modulo an integer number L of winding inside the U(1)). It follows that each value of the magnetic charge uniquely corresponds to an element of $\Pi_1(\tilde{H})$. The Dirac quantization condition is always satisfied in terms of the minimal charge [13, 14]. This statement can be shown to hold for all the monopoles corresponding to the symmetry breakings listed in the table 1[2] In section 4.2 we will study the case of the G_2 group in detail.

In the cases where G is not simply connected (e.g. in the 't Hooft - Polyakov solitonic solution $G = \mathrm{SO}(3) \to \mathrm{U}(1)$) we must exclude the non contractible paths inside G and this fact restricts the allowed values for the magnetic charge.

The one-to-one correspondence between magnetic charges and elements of $\pi_1(\tilde{H})$ allows to classify every topological configuration in terms of the magnetic charge which is defined in terms of the 't Hooft tensor (eq. (1.6), (1.7)). The explicit construction of the tensor will be the main goal of the next section.

4. The 't Hooft tensor

4.1 Construction

The 't Hooft tensor is a gauge invariant tensor which coincides with the residual abelian field strength in the unitary gauge. The magnetic field associated to the i-th monopole is

[2] We have checked this issue explicitly for the non exceptional groups and for G_2. For F_4, E_6, E_7 and E_8 it is a conjecture.

G	$H \times \mathrm{U}(1)$	λ_I	$\Pi_2(G/\tilde{H})$
$\mathrm{SU}(n)$	$\mathrm{SU}(n-m) \times \mathrm{SU}(m) \times \mathrm{U}(1)$	1	$\mathbf{Z}$
$\mathrm{SO}(2n+1)$	$\mathrm{SO}(2n-1) \times \mathrm{U}(1)$	1	$\mathbf{Z}$
$\mathrm{SO}(2n+1)$	$\mathrm{SO}(2m+1) \times \mathrm{SU}(n-m) \times \mathrm{U}(1)$	1,4	$\mathbf{Z}$
$\mathrm{SO}(2n+1)$	$\mathrm{SU}(n) \times \mathrm{U}(1)$	1,4	$\mathbf{Z}/Z_2$
$\mathrm{SO}(2n)$	$\mathrm{SO}(2n-2) \times \mathrm{U}(1)$	1	$\mathbf{Z}$
$\mathrm{SO}(2n)$	$\mathrm{SO}(2m) \times \mathrm{SU}(n-m) \times \mathrm{U}(1)$	1,4	$\mathbf{Z}$
$\mathrm{SO}(2n)$	$\mathrm{SU}(n-2) \times \mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{U}(1)$	1,4	$\mathbf{Z}/Z_2$
$\mathrm{SO}(2n)$	$\mathrm{SU}(n) \times \mathrm{U}(1)$	1	$\mathbf{Z}/Z_2$
$\mathrm{Sp}(2n)$	$\mathrm{Sp}(2m) \times \mathrm{SU}(n-m) \times \mathrm{U}(1)$	1,4	$\mathbf{Z}$
$\mathrm{Sp}(2n)$	$\mathrm{SU}(n-1) \times \mathrm{SU}(2) \times \mathrm{U}(1)$	1,4	$\mathbf{Z}$
$\mathrm{Sp}(2n)$	$\mathrm{SU}(n) \times \mathrm{U}(1)$	1	$\mathbf{Z}$
G_2	$\mathrm{SU}(2) \times \mathrm{U}(1)$	1,4,9	$\mathbf{Z}$
G_2	$\mathrm{SU}(2)' \times \mathrm{U}(1)$	1,4	$\mathbf{Z}$
F_4	$\mathrm{Sp}(6) \times \mathrm{U}(1)$	1,4	$\mathbf{Z}$
F_4	$\mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1)$	1,4,9	$\mathbf{Z}$
F_4	$\mathrm{SU}(3)' \times \mathrm{SU}(2)' \times \mathrm{U}(1)$	1,4,9,16	$\mathbf{Z}$
F_4	$\mathrm{Spin}(7) \times \mathrm{U}(1)$	1,4	$\mathbf{Z}$
E_6	$\mathrm{Spin}(10) \times \mathrm{U}(1)$	1	$\mathbf{Z}$
E_6	$\mathrm{SU}(5) \times \mathrm{SU}(2) \times \mathrm{U}(1)$	1,4	$\mathbf{Z}$
E_6	$\mathrm{SU}(6) \times \mathrm{U}(1)$	1,4	$\mathbf{Z}$
E_6	$\mathrm{SU}(3) \times \mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1)$	1,4,9	$\mathbf{Z}$
E_7	$\mathrm{Spin}(12) \times \mathrm{U}(1)$	1,4	$\mathbf{Z}$
E_7	$\mathrm{SU}(7) \times \mathrm{U}(1)$	1,4	$\mathbf{Z}$
E_7	$\mathrm{SU}(6) \times \mathrm{SU}(2) \times \mathrm{U}(1)$	1,4,9	$\mathbf{Z}$
E_7	$\mathrm{SU}(4) \times \mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1)$	1,4,9,16	$\mathbf{Z}$
E_7	$\mathrm{SU}(5) \times \mathrm{SU}(3) \times \mathrm{U}(1)$	1,4,9	$\mathbf{Z}$
E_7	$\mathrm{Spin}(10) \times \mathrm{SU}(2) \times \mathrm{U}(1)$	1,4	$\mathbf{Z}$
E_7	$E_6 \times \mathrm{U}(1)$	1	$\mathbf{Z}$
E_8	$\mathrm{Spin}(14) \times \mathrm{U}(1)$	1,4	$\mathbf{Z}$
E_8	$\mathrm{SU}(8) \times \mathrm{U}(1)$	1,4,9	$\mathbf{Z}$
E_8	$\mathrm{SU}(7) \times \mathrm{SU}(2) \times \mathrm{U}(1)$	1,4,9,16	$\mathbf{Z}$
E_8	$\mathrm{SU}(5) \times \mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1)$	1,4,9,16,25,36	$\mathbf{Z}$
E_8	$\mathrm{SU}(5) \times \mathrm{SU}(4) \times \mathrm{U}(1)$	1,4,9,16,25	$\mathbf{Z}$
E_8	$\mathrm{Spin}(10) \times \mathrm{SU}(3) \times \mathrm{U}(1)$	1,4,9,16	$\mathbf{Z}$
E_8	$E_6 \times \mathrm{SU}(2) \times \mathrm{U}(1)$	1,4,9	$\mathbf{Z}$
E_8	$E_7 \times \mathrm{U}(1)$	1,4	$\mathbf{Z}$

Table 1: Symmetry breaking of a generic compact group [First column] to the residual subgroup $\tilde{H} \times U(1)$ (modulo Z factors)[2nd column], the corresponding values of λ_I [third column] and the Homotopy group $\Pi_2(G/H)$. Notation: $Spin(N)$ indicates the covering group of $SO(N)$.

that of the group $\mathrm{U}(1)^i$ generated by T_3^i . We can define the e.m. field A_μ^i in terms of the gauge field A'_μ in the unitary gauge as:

$$A_\mu^i = Tr(\phi_0^i\, A'_\mu). \tag{4.1}$$

$\phi_0^i = \mu^i$, the fundamental weight (i $= 1, \ldots, r$), identifies the monopole species. If $b(x)$ is the gauge transformation bringing to a generic gauge and A_μ the transformed gauge field [15]

$$\begin{cases} A'_\mu = bA_\mu b^{-1} - \frac{i}{g}(\partial_\mu b)b^{-1} \\ \\ \phi_0^i = b\phi^i b^{-1} \end{cases} \tag{4.2}$$

the e.m. field can be written as:

$$A_\mu^i = Tr(\phi^i(A_\mu + \Omega_\mu)) \tag{4.3}$$

where $\Omega_\mu = -\frac{i}{g}\, b^{-1}\partial_\mu b$. We can rewrite the abelian field strength as

$$F_{\mu\nu}^i = Tr(\phi^i\, G_{\mu\nu}) + i\, g\, Tr(\phi^i\, [A_\mu + \Omega_\mu, A_\nu + \Omega_\nu]) \tag{4.4}$$

Because of the ciclycity of the trace only the part of $A_\mu + \Omega_\mu$ which does not belong to the invariance group of ϕ^i contributes. Indeed, denoting for the sake of simplicity as V_μ the vector $A_\mu + \Omega_\mu$,

$$Tr(\phi^i[V_\mu, V_\nu]) = Tr\,(V_\nu[\phi^i, V_\mu]) = Tr\,(V_\mu[V_\nu, \phi^i]) \tag{4.5}$$

To compute the second term in eq. (4.4) it proves convenient to introduce a projector P on the complement of the invariance algebra of ϕ^i . If we write V_μ as

$$V_\mu = \sum_{\vec{\alpha}} V_\mu^{\vec{\alpha}} E^{\vec{\alpha}} + \sum_j V_\mu^j H^j \tag{4.6}$$

where the sum on $\vec{\alpha}$ is extended to all positive and negative roots and the sum on j on all elements of Cartan algebra ($j = 1, \ldots, r$), we can certainly neglect the last term, which commutes with ϕ^i. Moreover the generic $E^{\vec{\alpha}}$ is part of the little group of ϕ^i whenever

$$[\phi^i, E^{\vec{\alpha}}] = (\vec{c}^{\,i} \cdot \vec{\alpha})\, E^{\vec{\alpha}} = 0 \tag{4.7}$$

If instead $(\vec{c}^{\,i} \cdot \vec{\alpha}) \neq 0$, $E^{\vec{\alpha}}$ belongs to the complement. It is trivial to verify that projection on the complement $P^i\, V_\mu$ is given by

$$P^i V_\mu = 1 - \prod_{\vec{\alpha}}{}' \left(1 - \frac{[\phi^i, [\phi^i, \ ]]}{(\vec{c}^{\,i} \cdot \vec{\alpha})^2}\right) V_\mu \tag{4.8}$$

where $[\phi^i, \ \ ]\, V_\mu = [\phi, V_\mu]$ and the product $\prod'_{\vec{\alpha}}$ runs on the roots $\vec{\alpha}$ such that $\vec{c}^{\,i} \cdot \vec{\alpha} \neq 0$ and only one representative is taken of the set of the roots having the same value of $\vec{c}^{\,i} \cdot \vec{\alpha}$.

Indeed if any element $E^{\vec{\alpha}}$ in eq. (4.6) commutes with ϕ^i, $P^i E^{\vec{\alpha}} = (1-1)E^{\vec{\alpha}} = 0$. If for any $E^{\vec{\alpha}}$

$$[\phi^i, E^{\vec{\alpha}}] = (\vec{c}^{\,i} \cdot \vec{\alpha})E^{\vec{\alpha}} \qquad (\vec{c}^{\,i} \cdot \vec{\alpha}) \neq 0 \tag{4.9}$$

JHEP10(2008)096

one of the factors $\left(1-\frac{[\phi^i,[\phi^i\ ]]}{(\vec{c}^{\,i}\cdot\vec{\alpha})^2}\right)$ in the definition eq. (4.8) will give zero and $PE^{\vec{\alpha}}=E^{\vec{\alpha}}$. In order to simplify the notation we denote by λ^i_I the different non zero values which $(\vec{c}^{\,i}\cdot\vec{\alpha})^2$ can assume and rewrite P^iV_μ as

$$P^iV_\mu = 1-\prod_I\left(1-\frac{[\phi^i,[\phi^i,\ ]]}{\lambda^i_I}\right)V_\mu \tag{4.10}$$

Eq. (4.4) can be rewritten as

$$F^i_{\mu\nu}=Tr(\phi^i\,G_{\mu\nu})+igTr(\phi^i\,[P^i\,(A_\mu+\Omega_\mu)\,,A_\nu+\Omega_\nu]) \tag{4.11}$$

For our purpose it is sufficient to project only one of the operators in the commutator. By use of eq. (4.10) and recalling that

$$D_\mu\phi^i=-ig[A_\mu+\Omega_\mu\,,\,\phi^i\,] \tag{4.12}$$

the generalized 't Hooft tensor reads as

$$\begin{aligned}F^i_{\mu\nu} &= Tr(\phi^iG_{\mu\nu})-\frac{i}{g}\sum_I\frac{1}{\lambda^i_I}\,Tr\left(\phi^i[D_\mu\phi^i,D_\nu\phi^i]\right)+\\ &+\frac{i}{g}\sum_{I\neq J}\frac{1}{\lambda^i_I\lambda^i_J}\,Tr\left(\phi^i[[D_\mu\phi^i,\phi^i],[D_\nu\phi^i,\phi^i]]\right)+\cdots\end{aligned} \tag{4.13}$$

To summarize, we have to compute for each root $\vec{\alpha}$ the (known) commutator $[\phi^i,E^{\vec{\alpha}}]=(\vec{c}^{\,i}\cdot\vec{\alpha})\,E^{\vec{\alpha}}$, where ϕ^i are the fundamental weights associated to each simple root. This will give us the set of the values of λ^i_I to insert into eq. (4.13). For SU(N) group $[\phi^i,E_{\vec{\alpha}}]=(\vec{c}^{\,i}\cdot\vec{\alpha})E_{\vec{\alpha}}$, where $(\vec{c}^{\,i}\cdot\vec{\alpha})=0,\pm1$, so the projector is simply

$$P^iV_\mu=[\phi^i,[\phi^i,V_\mu]] \tag{4.14}$$

and the 't Hooft tensor is the usual one

$$F^i_{\mu\nu}=Tr(\phi^iG_{\mu\nu})-\frac{i}{g}Tr(\phi^i[D_\mu\phi^i,D_\nu\phi^i]) \tag{4.15}$$

For a generic group the projector is more complicated and it can depend on the root chosen. Results are listed in table 1.

4.2 't Hooft tensors for G_2

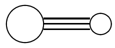
We now specialize the above results to the case of gauge group G_2. It is natural to view G_2 as a subgroup of SO(7) [6]. In fact G_2 is the subgroup of the 7×7 orthogonal matrices Ω which satisfy the relations

$$T_{abc}=T_{def}\Omega_{da}\Omega_{eb}\Omega_{fc} \tag{4.16}$$

T_{abc} is a totally antisymmetric tensor whose non-zero elements are given by

$$T_{127}=T_{154}=T_{235}=T_{264}=T_{374}=T_{576}=1$$

According to section 2, we consider the breaking of G_2 to a subgroup SU(2)$\times$U(1). Dynkin diagram of G_2 is depicted as follow



where the first circle corresponds to the longest simple root e_1 and the second one to the other e_2. The residual invariance group is obtained by erasing one of the two roots in turn. It's Dynkin diagram consists of one single circle, which means $H = \mathrm{SU}(2)$. The explicit form of the generators of these residual SU(2) subgroups is, in the notation of [6],

$$\begin{aligned}
T_+^{(1)} &= (|1\rangle\langle 2| - |5\rangle\langle 4|) \qquad\qquad T_-^{(1)} = (|2\rangle\langle 1| - |4\rangle\langle 5|) \\
T_3^{(1)} &= (\,|1\rangle\langle 1| - |2\rangle\langle 2| - |4\rangle\langle 4| + |5\rangle\langle 5|) \\
T_+^{(2)} &= |3\rangle\langle 5| - |2\rangle\langle 6| - \sqrt{2}|7\rangle\langle 1| - \sqrt{2}|4\rangle\langle 7| \\
T_-^{(2)} &= |5\rangle\langle 3| - |6\rangle\langle 2| - \sqrt{2}|1\rangle\langle 7| - \sqrt{2}|7\rangle\langle 4| \\
T_3^{(2)} &= -2|1\rangle\langle 1| + |2\rangle\langle 2| + |3\rangle\langle 3| + 2|4\rangle\langle 4| - |5\rangle\langle 5| - |6\rangle\langle 6|
\end{aligned}$$

- If we break the simple root e_1 we have as little group $\mathrm{SU}(2) \times \mathrm{U}(1)$ and the corresponding maximal weight reads

$$\phi_0^{(1)} = \mathrm{diag}(0, -1, 1, 0, 1, -1, 0) \tag{4.17}$$

The coefficients $(\lambda_I^{(1)})$ are equal to $1, 4$. By using eq. (4.13) 't Hooft tensor reads

$$\begin{aligned}
F_{\mu\nu}^{(1)} &= Tr(\phi^{(1)} G_{\mu\nu}) - \frac{5i}{4g} Tr\left(\phi^{(1)}[D_\mu \phi^{(1)}, D_\nu \phi^{(1)}]\right) + \\
&\quad + \frac{i}{4g} Tr\left(\phi^{(1)}[[D_\mu \phi^{(1)}, \phi^{(1)}], [D_\nu \phi^{(1)}, \phi^{(1)}]]\right)
\end{aligned} \tag{4.18}$$

More precisely the invariance subgroup is $\frac{\mathrm{SU}(2)\times \mathrm{U}(1)}{Z_2}$. Indeed, if we write $T_3^{(1)}$ as

$$T_3^{(1)} = \mathrm{diag}(1, -1, 0, -1, 1, 0, 0) = \frac{\phi_0^{(1)}}{2} + h \tag{4.19}$$

where h is

$$h = \mathrm{diag}(1, -1/2, -1/2, -1, 1/2, 1/2, 0) \tag{4.20}$$

we can see that, when magnetic charge are even integers, the corresponding loops wind only in the U(1), while for odd integers the loops travel partly in U(1), from identity to the non-trivial element of the center of SU(2), and the rest in the non-abelian SU(2) subgroup.

- If we break the other simple root e_2 we have as little group $\mathrm{SU}(2)' \times \mathrm{U}(1)$ and the correspondent maximal weight reads

$$\phi_0^{(2)} = \mathrm{diag}(-1, -1, 2, 1, 1, -2, 0) \tag{4.21}$$

with $(\lambda_I^{(2)}) = 1, 4, 9$. These values of coefficients give us a 't Hooft tensor of the form

$$F_{\mu\nu}^2 = Tr(\phi^{(2)} G_{\mu\nu}) - \frac{49i}{36g} Tr\left(\phi^{(2)}[D_\mu \phi^{(2)}, D_\nu \phi^{(2)}]\right) +$$

$$+\frac{7i}{18g}\,Tr\left(\phi^{(2)}[[D_\mu\phi^{(2)},\phi^{(2)}],[D_\nu\phi^{(2)},\phi^{(2)}]]\right)$$
$$-\frac{i}{36g}\,Tr\left(\phi^{(2)}[[[D_\mu\phi^{(2)},\phi^{(2)}],\phi^{(2)}],[[D_\nu\phi^{(2)},\phi^{(2)}],\phi^{(2)}]]\right) \tag{4.22}$$

Similarly to the previous case the residual gauge group is $\frac{\mathrm{SU}(2)\times\mathrm{U}(1)}{Z_2}$ and for even charges loops wind only on U(1), while for odd charges loops run partly in U(1) and the rest in SU(2).

5. Discussion

The experimental limits on the observation of free quarks in nature indicate that confinement is an absolute property, in the sense that the number of free quarks is strictly zero due to some symmetry. Deconfinement is a change of symmetry. Since color is an exact symmetry, the only way to have an extra symmetry, which can be broken, is to look for a dual description of QCD. The extra degrees of freedom are infrared modes related to boundary conditions. This is a special case of the so called geometric Langlands program of ref. [1].

The relevant homotopy in 3+1 dimensions is a mapping of the two dimensional sphere S_2 at spatial infinity onto the group. The homotopy group is thus Π_2, configurations are monopoles [2, 3] and the quantum numbers magnetic charges.

For a generic gauge group of rank r there exist r different magnetic charges Q^a labelling the dual states. The existence of magnetic charges implies a violation of Bianchi identities by the abelian gauge field coupled to them. The gauge invariant abelian field strength coupled to Q^a is known as 't Hooft tensor. In this paper we analyzed monopoles in a generic compact gauge group and we explicitly constructed the corresponding 't Hooft tensor.

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JHEP10(2008)096